

## Possible Statistics of Scale Invariant Systems

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(Received 28 April 1995, revised 11 January 1996, accepted 1 February 1996)

PACS.11.30.-j – Symmetry and conservation laws

**Abstract.** — A relativity postulate states the equivalence of rationalized systems of units, constructed as power laws of the scale  $\ell$ . In a scale invariant system, described by a random physical field  $\phi$ , this relativity selects the set of similarity transformations coupling  $\ell$  and  $\phi$ . Acceptable transformations are classified into six possible groups, according to two dimensionless parameters: an exponent  $C$  characteristic of the physical system, and  $\Lambda$  describing the small scale / large scale symmetry breaking. Symmetry severely constrains the successive moments of  $\phi$ , and hence the shape of its probability distribution. For instance, the Newtonian case  $C/\Lambda \rightarrow \infty$  corresponds to self-similar statistics, the ultra-relativistic case  $C/\Lambda \rightarrow 0$  to deterministic fields, and the case  $\Lambda = 1$  to a log-Poisson statistics. These cases are applied to hydrodynamical turbulence in the companion paper.

**Résumé.** — Un postulat de relativité établit l'équivalence des systèmes d'unités rationalisés, qui sont construits comme des lois de puissance de l'échelle  $\ell$ . Dans un système invariant d'échelle, décrit par un champ physique aléatoire  $\phi$ , cette relativité sélectionne l'ensemble des transformations de similarité qui couple  $\ell$  et  $\phi$ . Les transformations acceptables sont classées en six groupes possibles par deux paramètres sans dimension: un exposant  $C$  caractéristique du système physique, et  $\Lambda$  qui décrit la brisure de symétrie entre petites et grandes échelles. La symétrie contraint fortement les moments successifs de  $\phi$ , et donc la forme de sa distribution de probabilité. En particulier, le cas newtonien  $C/\Lambda \rightarrow \infty$  correspond à une statistique self-similaire, le cas ultra-relativiste  $C/\Lambda \rightarrow 0$  à un champ déterministe, et le cas  $\Lambda = 1$  à une statistique log-Poissonnienne. Ces cas sont appliqués à la turbulence hydrodynamique dans l'article joint.

### 1. Introduction

Scale invariance usually refers to systems conserving the same properties or shape at different scales. The most famous examples are fractal sets [1]. Strictly speaking, only infinite size

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systems can be fully scale invariant. This is the case considered in the present paper. Breaking of the exact scale symmetry by introduction of a minimum and/or maximum scale will be discussed briefly in Section 5.1.

Consider then a field  $\phi(\mathbf{x})$  measuring a physical characteristic, such as e.g. a velocity or a density field, of a scale invariant system. Then  $\phi$  is invariant under a family of dilations  $\mathcal{S}_h(\lambda) : \mathbf{x} \rightarrow \lambda\mathbf{x}; \phi \rightarrow \lambda^h\phi$ , for any  $\lambda$  and  $h$ . Equivalently, we will define below the field  $\phi_\ell(\mathbf{x})$  at scale  $\ell$ . This symmetry then appears as invariance by arbitrary change of units for  $\ell$  and  $\phi_\ell$ :  $\ell \rightarrow \lambda\ell; \phi_\ell \rightarrow \lambda^h\phi_\ell$ . For a random homogeneous scale invariant field, the moments of  $\phi_\ell(\mathbf{x})$  are power laws of  $\ell$ :

$$\langle \phi_\ell(x)^n \rangle \propto \ell^{\zeta(n)}, \quad (1)$$

where  $\zeta(n)$  is a concave function of  $n$  (see [2] and appendix E).

Our goal is to show that the scale symmetry only is sufficient to impose strong constraints on statistics of scale invariant systems. We explore the consequences on a scale invariant system, of a postulate, analogous to the equivalence principle of special relativity. This postulate, stated in Section 2, generalizes the existence of equivalent systems of units, by defining equivalent “reference fields”. When combined with general properties stemming from the scale invariance, the postulate yields generic constraints on the statistics of random processes in scale invariant systems and selects the possible shape for the exponents of the moments,  $\zeta(n)$  (Eq. (1)).

We classify all possible models using two parameters (Sect. 4): an exponent  $C$  characteristic of the physical system, and a large scales / small scales symmetry-breaking parameter  $\Lambda$ . The results are applied in [3] to the special problem of hydrodynamic turbulence, which motivated the present article. The analogy with special relativity was inspired by Nottale and Pocheau although there are many differences with their work [4–6]; in a forthcoming paper we will examine in more details this analogy, and extend it to general relativity by letting  $\lambda$  vary with the scale [7].

## 2. From Units to Reference Fields

**2.1. UNITS AND EQUIVALENCE OF UNITS SYSTEMS.** — Our results are based on a natural generalization of the notion of units. A measure of a physical quantity is a dimensionless number multiplied by a unit. Accordingly, units are usually chosen according to the phenomenon observed, to deal with numbers of order one: for instance, astronomers use parsec to describe distances, while nuclear physicists use Fermis. This freedom is allowed by an intuitive basic postulate: all systems of units are equivalent to describe the laws of physics.

A few universal laws require to compare measurements at different scales: to check his law of gravitation, Newton had to go through conversions between his earthly units and astronomical units. Such operation is of course simpler in rationalized systems, in which successive units are in a constant ratio: whether decimal systems like the MKS or the CGS, or duodecimal, or binary, for instance. Physics itself is of course independent of our choices: planet trajectories around the Sun are ellipses anyway, but using kilometers-meters-centimeters makes the physicist’s life easier than miles-yard-inches to demonstrate it.

### 2.2. A REFERENCE FIELD FOR MEASURING FIELDS

**2.2.1. Definition.** — We want to generalize what precedes to the case of a random field  $\phi(\mathbf{x})$ , involving many different scales of length which may or not affect each other. Note that what follows will be trivially true for deterministic fields. There are many equivalent ways to define the “field at scale  $\ell$ ”. If  $\phi$  is regular enough, for instance, we can simply derive a coarse-grained

field  $\phi_\ell$  through a spatial average on a sphere:

$$\phi_\ell(\mathbf{x}) = \frac{1}{\ell^D} \int_{|\mathbf{y}| < \ell} \phi(\mathbf{x} - \mathbf{y}) d^D \mathbf{y}. \quad (2)$$

From now on, we consider for convenience only positive fields, by dividing, if needed, the field into its positive and negative parts:

$$\phi_\ell(\mathbf{x}) = \phi_\ell^{(+)}(\mathbf{x}) - \phi_\ell^{(-)}(\mathbf{x}), \quad (3)$$

where the  $\phi_\ell^{(\pm)}$  are positive and treated separately.

At each point  $\mathbf{x}$  and at each scale  $\ell$ , we can select a unit well suited to measure  $\phi_\ell(\mathbf{x})$ . The set  $\mathcal{R}_\ell(\mathbf{x})$  of all these units is itself a field, and we call it a “reference field”. If the field is random, so can be its reference field (see e.g. Sect. 5.2 for an interpretation within the context of multifractal theory). It has the same physical dimension as  $\phi$ , e.g. a velocity if  $\phi$  is a velocity field, and is also positive. The result of the measurement of  $\phi_\ell$  is then expressed through the ratio  $\phi_\ell/\mathcal{R}_\ell$ , or, for an homogeneous field measured with an homogeneous reference field, through the average  $\langle \phi_\ell/\mathcal{R}_\ell \rangle$  on the realization of this ratio, which only depends on the scale  $\ell$ .

*2.2.2. A Special Class of Reference Fields.* — Being more general than the simple notion of unit, the set of all possible reference fields has no reason to obey the equivalence principle satisfied by the units. There is an exception for a subset of the reference fields, namely homogeneous scale invariant fields. In that case, measurements performed at one scale are physically equivalent to measurements performed at any other scale up to a scale invariant factor. This set of reference fields, associated to a power law of the scale  $\ell$ , then appears as a generalization of the rationalized unit systems discussed in Section 2.1. Then, in the same way that binary systems are equivalent to decimal systems, we may postulate that all homogeneous scale invariant reference fields characterized by their associated power laws (Eq. (1)) are physically equivalent. This extrapolation is the basis of our postulate, stated in Section 3.1.

*2.2.3. Examples of Homogeneous Scale Invariant Reference Fields.* — Consider a scale invariant system, with a physical property described by an homogeneous field  $\phi(\mathbf{x})$ . One may generate an arbitrary number of homogeneous scale invariant reference fields. Indeed, since  $\phi(\mathbf{x})$  is scale invariant, all moments of  $\phi_\ell(\mathbf{x})$  are power laws of  $\ell$ . Then, the maximal value of experimental realizations, defined e.g. as:

$$\phi_{\max}(\ell) = \lim_{n \rightarrow \infty} \frac{\langle \phi_\ell^{n+1} \rangle}{\langle \phi_\ell^n \rangle}, \quad (4)$$

is also a power law of  $\ell$ :

$$\phi_{\max}(\ell) \propto \ell^\Delta, \quad (5)$$

where  $\Delta$  is a characteristic exponent. Being independent on  $\mathbf{x}$ ,  $\phi_{\max}$  is trivially homogeneous.

It is then easy to check that the family of fields  $\{\phi_\ell^{1-\alpha} \phi_{\max}^\alpha, 0 \leq \alpha \leq 1\}$ , is composed of homogeneous, scale invariant fields with same physical dimension as  $\phi$ . Any member of this family can then be used equivalently to measure  $\phi_\ell$ , or any field with same dimension as  $\phi$ .

**2.3. CONVENIENT NOTATIONS.** — To make use of what precedes, we introduce new notations dedicated to our argumentation. More precisely, rather than multiplicative constants on  $\ell$  and  $\phi$ , we prefer to deal with their logarithms and thus turn to additive constants.

In the literature, an usual way to define the scales is to provide a discrete slicing in the scale space. The  $n^{\text{th}}$  scale is then  $\ell_n = \ell_0 K^n$ ; here  $n$  is an integer number,  $\ell_0$  the unit chosen for the

scales, and  $K$  the chosen resolution. Here we generalize  $n$  as a continuous variable, the real number  $T$ :

$$T = \ln \left( \frac{\ell}{\ell_0} \right), \quad (6)$$

and similarly :

$$X(T) = \ln \left\langle \frac{\phi_\ell}{\mathcal{R}_\ell} \right\rangle, \quad (7)$$

where  $\langle \rangle$  denotes an average on realizations.

In the neighbourhood of a given scale, and for homogeneous fields, we can define a local exponent

$$\frac{dX}{dT} = \frac{d \ln \langle \phi / \mathcal{R}_\ell \rangle}{d \ln(\ell / \ell_0)} \quad (8)$$

Of course, we may also measure in a similar way the local scale variations of any power of  $\phi$ , which is also an homogeneous field. When the measured field is scale invariant, with moments following (1), and is measured using the reference field  $\phi^{1-\alpha} \phi_{\max}^\alpha$ , the local exponent becomes constant and takes a simple expression:

$$\begin{aligned} \frac{dX}{dT} &= \frac{d \ln \langle \phi / \phi^{1-\alpha} \phi_{\max}^\alpha \rangle}{d \ln(\ell / \ell_0)} \\ &= -\alpha \Delta + \zeta(\alpha). \end{aligned} \quad (9)$$

We call this quantity “the intermittency function of  $\phi$ ”. Since it is a function of  $\alpha$  only, we note it  $\delta\zeta(\alpha)$ :

$$\zeta(\alpha) = \alpha \Delta + \delta\zeta(\alpha). \quad (10)$$

### 3. Axiomatic Derivation of the Similarity Transformation

Our goal is now to derive the possible similarity transformations, using measurements of a scale invariant field with respect to two different reference fields. This requires only two simple postulates, stated in the next section, expressing the scale invariance and the equivalence of reference fields. To allow a linear reading of the axiomatic derivation of the similarity transformations, we reject all computations in appendix.

**3.1. POSTULATES.** — Our notations turn particularly useful to express simply the scale invariance and the equivalence of reference fields: Postulate 1 (scale invariance): *The log-coordinates  $(X, T)$  are invariant under global translation*, i.e. under an arbitrary choice of their origin  $(X_0, T_0)$ : the  $(X, T)$  coordinate space is homogeneous. Postulate 2 (equivalence): *Among all imaginable reference fields, there exists a continuous class of equivalent reference fields built on scale invariant fields*. When describing a given field in two different reference field  $\mathcal{R}$  and  $\mathcal{R}'$ , one must link  $(X, T)$  and  $(X', T')$ . Going from one reference field to another involves only one number characteristic of their relative variation  $V_{\mathcal{R}|\mathcal{R}'}$ ; this number is the exponent corresponding to the measure of the field  $\mathcal{R}$  with respect to the reference  $\mathcal{R}'$ . The set of such similarity transformations thus forms a (at least semi-)group.

**3.2. GROUP STRUCTURE.** — These symmetry considerations constitute the cornerstone of the analogy with mechanics exposed in paper [7]. To exploit them, we closely follow in Appendix the very pedagogical derivation of special relativity given by Levy-Leblond [8] and Notale [4], with suitable adaptation of the notations, and determine the class of possible reference

transformations. The main points are demonstrated in appendix:

- postulate 1 implies that transformations are linear;
- since nothing forbids a coupling between field and scale, their most general representation is a  $2 \times 2$  matrix coupling intervals in  $X$  and  $T$ ;
- if scale symmetry is continuous these matrices are real [9];
- postulate 2 implies that their determinant is 1.
- thus there are 3 free parameters. We arbitrarily choose the corresponding notations as follow, with no loss of generality. The first one labels the particular transformation; it can be shown to be equal to the variation of the second reference field to the first one  $V_{\mathcal{R}'|\mathcal{R}}$ . The two remaining ones characterize the group associated to the physical system: the second is noted  $C$  to maintain analogy with usual Lorentz notations, the third is noted  $\Lambda$ . Omitting the indices on  $V_{\mathcal{R}'|\mathcal{R}}$  for simplification, the transformation then writes:

$$\begin{pmatrix} CT' \\ X' \end{pmatrix} = \Gamma(V) \begin{pmatrix} 1 - 2\Lambda V/C & (\Lambda^2 - 1)V/C \\ -V/C & 1 \end{pmatrix} \begin{pmatrix} CT \\ X \end{pmatrix}, \tag{11}$$

where  $\Gamma(V)$  ensures a determinant 1:

$$\Gamma(V)^{-2} = 1 - 2\Lambda \frac{V}{C} + (\Lambda^2 - 1) \frac{V^2}{C^2}. \tag{12}$$

Composing two transformations between reference fields  $\mathcal{R}$  and  $\mathcal{R}'$ , then  $\mathcal{R}'$  and  $\mathcal{R}''$ , leads to a third transformation between  $\mathcal{R}$  and  $\mathcal{R}''$ , labelled with

$$V_{\mathcal{R}''|\mathcal{R}} = V_{\mathcal{R}''|\mathcal{R}'} \otimes V_{\mathcal{R}'|\mathcal{R}}, \tag{13}$$

where  $\otimes$  is the composition law:

$$V \otimes V' = \frac{V + V' - 2\Lambda VV'/C}{1 - (\Lambda^2 - 1)VV'/C^2}. \tag{14}$$

Although we assumed only a semi-group structure, it actually turns out to be a group, which is moreover commutative: each element has an opposite  $V^{[-1]} = -V/(1 - 2\Lambda V/C)$ , where we note  $V^{[p]} = V \otimes V \otimes \dots \otimes V$  ( $p$  times). Here  $V = 0$  is the neutral element.

3.3. INTERPRETATION OF PARAMETERS. — Our notations can now be interpreted:

- $C$  is an exponent, typical of the average of the fixed points for the composition law  $\otimes$ :

$$C_+ = \frac{C}{\Lambda - 1} \quad ; \quad C_- = \frac{C}{\Lambda + 1}. \tag{15}$$

To identify our formalism with physical quantities, we state Postulate 3: *these fixed points  $C_{\pm}$  are the codimension of the most intermittent structures in the system.* Justification of this postulate comes from an identification with the mono-fractal case, see Section 4.2. There can be zero, one or two such codimensions.

- $\Lambda$  characterizes the symmetry-breaking between small scales and large scales. In fact, in the case  $\Lambda = 0$ , composition laws would be invariant under  $X \rightarrow -X, V \rightarrow -V, T \rightarrow -T$ , i.e.  $\ell \rightarrow 1/\ell$ . Note that adding this symmetry of course selects the Lorentz composition law [7]. Similarly,  $C/\Lambda \rightarrow \infty$  selects the Newton composition law.
- Within continuous, exact scale invariance, codimensions are positive, real numbers, smaller than the space dimension  $D$  (see however [10] for codimension larger than  $D$ ). Requiring positive values of the invariant codimension(s) selects:

$$C \geq 0 \quad \text{and} \quad \Lambda \geq 1, \tag{16}$$

or equivalently:

$$0 \leq C_- \leq C_+. \quad (17)$$

Here we have used the restriction (B.8) allowed by the symmetries.

3.4. MOMENT RELATION. — The transformation (11) can be used to derive a relation between successive moments of  $\phi_\ell$ . Indeed, measuring  $\phi_\ell$  with respect to  $\phi^{1-p}\phi_{\max}^p$  and  $\phi^{1-q}\phi_{\max}^q$ , writing the corresponding transformation using the same origins (see Eq. (A.4)) and going back into physical variable  $\ell$  and  $\phi$ , we get:

$$\begin{aligned} \frac{\ell'}{\ell_0} &= \left\langle \frac{\phi^q}{\phi_{\max}^q} \right\rangle^{[\Lambda^2-1]\Gamma\delta\zeta/C^2} \times \left( \frac{\ell}{\ell_0} \right)^{[1-2\Lambda\delta\zeta/C]\Gamma}, \\ \left\langle \frac{\phi^p}{\phi_{\max}^p} \right\rangle &= \left\langle \frac{\phi^q}{\phi_{\max}^q} \right\rangle^\Gamma \times \left( \frac{\ell}{\ell_0} \right)^{-\delta\zeta\Gamma}, \end{aligned} \quad (18)$$

where we note  $\delta\zeta = \delta\zeta(q-p)$  and, according to (12):

$$\Gamma(\delta\zeta) = \frac{1}{\sqrt{\left(1 - \frac{\delta\zeta}{C_-}\right) \left(1 - \frac{\delta\zeta}{C_+}\right)}}. \quad (19)$$

This transformation is the main point of the present paper, and links successive moments of statistical quantities in any scale covariant problem. This coupling between the moments of a physical quantity and the length scale is difficult to understand intuitively. In the generic case, however, there is no reason for these exponents to be independent. This is a powerful formalism as we now demonstrate by examining its predictions.

## 4. Consequences

### 4.1. INTERMITTENCY FUNCTION

4.1.1. *Composition.* — The composition law between reference fields (13) can be used to derive an analog composition law for the intermittency function. Indeed, it may be checked that the variation of the reference field  $\phi^{1-\alpha}\phi_{\max}^\alpha$  with respect to the reference field  $\phi^{1-\beta}\phi_{\max}^\beta$  is simply  $\delta\zeta(\beta-\alpha)$ . Composing then two successive similarity transformations, we then get from (13):

$$\delta\zeta(\beta-\alpha) = \delta\zeta(\beta-\gamma) \otimes \delta\zeta(\gamma-\alpha), \quad (20)$$

for any  $\alpha, \beta, \gamma$ .

4.1.2. *Values.* — The shape of the composition law (20) selects the possible intermittency functions  $\delta\zeta(p)$  in a scale invariant system. They depend only on three parameters: the two invariant codimensions  $C_+$  and  $C_-$ , and the value in one point other than 0, say  $\delta\zeta(1)$ .

Applying equation (20) recursively with  $\alpha = 0$ ,  $\beta = n$  and  $\gamma = n-1$ , provides the value of the function  $\delta\zeta$  on integers:

$$\delta\zeta(n) = \left( \delta\zeta(1) \right)^{[n]}. \quad (21)$$

The shape of the corresponding function depends on the invariant codimensions. In order to ensure consistency with earlier notations [11], we introduce the auxiliary function  $\beta$ , transforming

the composition law (20) into a multiplication. The three cases are:

- for  $C_+$  and  $C_-$  real and different (case  $\Lambda^2 \neq 1$ ), we have:

$$\begin{aligned} \delta\zeta(n) &= C_+ C_- \frac{1 - \beta(1)^n}{C_+ - C_- \beta(1)^n}, \\ \beta(1) &= \frac{C_+ \delta\zeta(1) - C_-}{\delta\zeta(1) - C_+}. \end{aligned} \tag{22}$$

Note the divergency occurring at

$$n_* = \frac{\ln(C_+/C_-)}{\ln \beta(1)}. \tag{23}$$

- for  $C_+ = C_- = C_0$  real (case  $C, \Lambda \rightarrow \infty, C/\Lambda = C_0$ ), we have:

$$\begin{aligned} \delta\zeta(n) &= \frac{n C_0 \delta\zeta(1)}{C_0 + (n - 1)\delta\zeta(1)}, \\ \beta(1) &= 1. \end{aligned} \tag{24}$$

Here, the divergency occurs at

$$n_* = \frac{C_0 - \delta\zeta(1)}{\delta\zeta(1)}. \tag{25}$$

- for  $C_+ = \infty; C_- = C/2$  real (case  $\Lambda = 1$ ) we have:

$$\begin{aligned} \delta\zeta(n) &= \delta\zeta(1) \frac{1 - \beta(1)^n}{1 - \beta(1)}, \\ \beta(1) &= 1 - \frac{\delta\zeta(1)}{C_-}. \end{aligned} \tag{26}$$

Same, for  $C_- = \infty; C_+ = -C/2$  real (case  $\Lambda = -1$ ),  $\beta(1) = 1 - \delta\zeta(1)/C_+$ . These cases present no divergency.

#### 4.1.3. Comments

- Note that the special cases  $\delta\zeta(n) = 0, \delta\zeta(n) = C_{\pm}, \delta\zeta(n) = n\delta\zeta(1)$  appear as limits of the preceding cases.
- Generalization to  $n$  non integer is straightforward: first for  $n$  rational, by using the replacement  $\delta\zeta(n) = \delta\zeta(p \times n/p)$  for any  $p$  integer; then to  $n$  real by continuity.
- Note that in all cases, the function  $\beta$  obeys:

$$\beta(n) = \beta(1)^n, \tag{27}$$

so that the function  $\beta$  is an exponential. However, since the intermittency function is real, only cases corresponding to  $\beta(1) \geq 0$  can be considered.

- This constrains the possible values of  $\delta\zeta(1)$ :

$$\begin{aligned} C_- \leq \delta\zeta(1) \leq C_+, & \quad \text{for } C_- C_+ < 0, \\ \frac{\delta\zeta(1)}{C_-} \leq 1 \text{ or } \frac{\delta\zeta(1)}{C_+} \geq 1, & \quad \text{for } C_- C_+ > 0. \end{aligned} \tag{28}$$

- The shape of the codimension function in the generic, degenerate or log-Poisson cases is given in Figure 1 for typical values of the parameters. Note that in all cases,  $\delta\zeta(n) \rightarrow C_-$  or  $C_+$  when  $n \rightarrow -$  or  $+\infty$ .

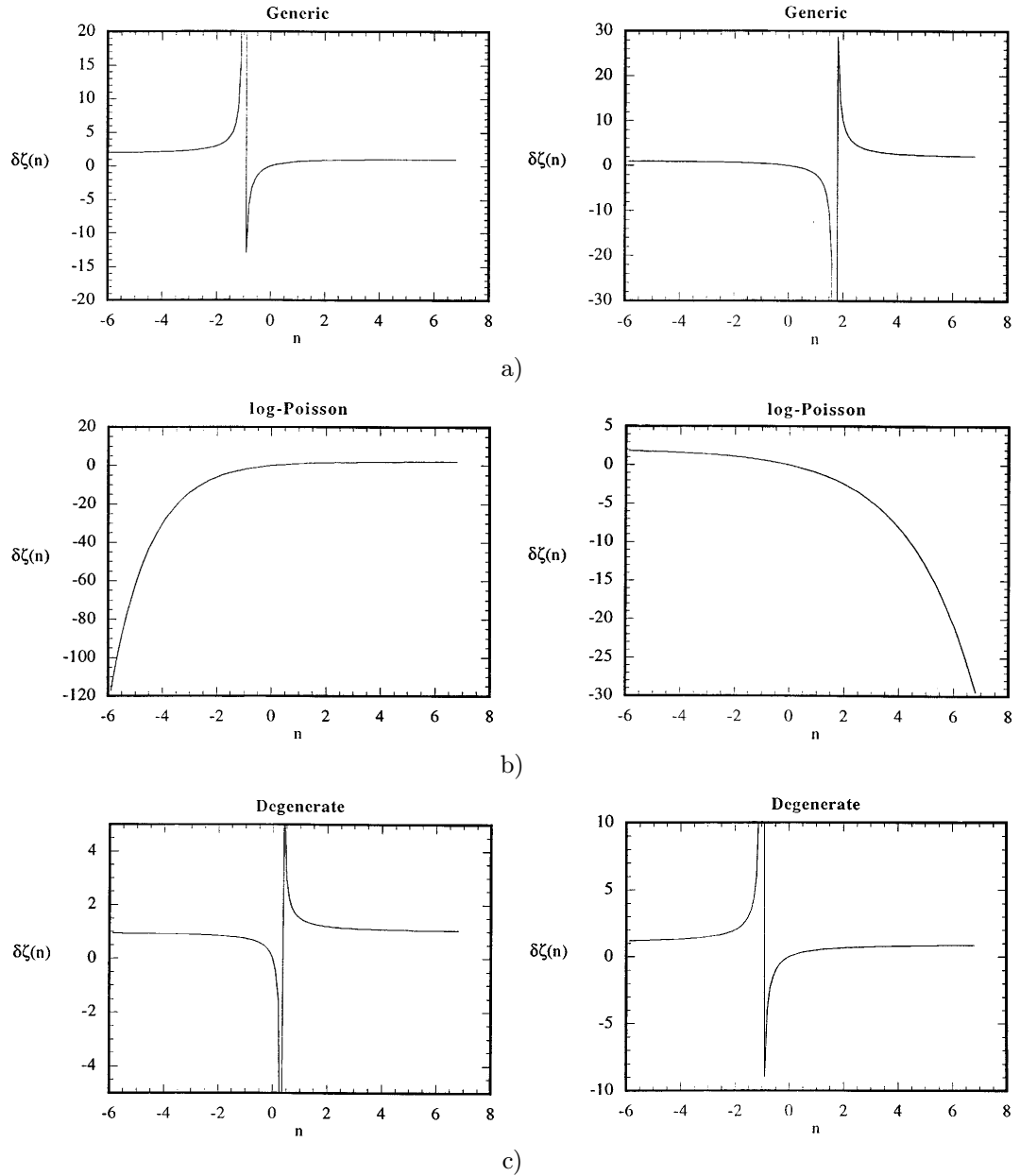


Fig. 1. — Typical shape of the intermittency function as a function of the parameters  $C_{\pm}$  and  $\delta\zeta(1)$ . Only the part in which the function is concave is physically admissible. a) Generic case  $C_+ = 2$ ;  $C_- = 1$  for two values of  $\delta\zeta(1)$  corresponding to  $\beta(1) = 1/2$  (top) or  $\beta(1) = 3/2$  (bottom). Note the change of concavity and the divergence occurring at  $n_*$  in both cases. The intermittency function tends towards  $C_{\pm}$  for  $n \rightarrow \mp\infty$  in the first case, and for  $n \rightarrow \pm\infty$  in the second case. b) Log-Poisson case  $C_+ = \infty$ ;  $C_- = 2$  for two values of  $\delta\zeta(1)$  corresponding to  $\beta(1) = 1/2$  (top) or  $\beta(1) = 3/2$  (bottom). The intermittency function tends towards  $C_-$  for  $n \rightarrow \infty$  in the first case, and for  $n \rightarrow -\infty$  in the second case. c) Degenerate case  $C_+ = C_- = C_0 = 1$ ; for two values of  $\delta\zeta(1)$  corresponding to  $n_* = 1/3$  (top) or  $n_* = -1$  (bottom). Note the change of concavity and the divergence occurring at  $n_*$  in both cases. The intermittency function tends towards  $C_0$  for  $n \rightarrow \pm\infty$  in both cases.



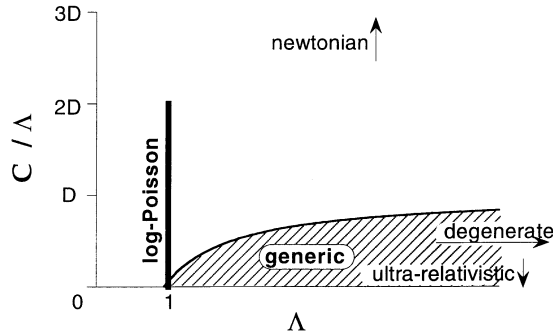


Fig. 2. — The two disjoint domains for possible values of  $C$  as a function of  $\Lambda$ . The generic domain is for  $\Lambda > 1$ , and is limited by  $0 < C/\Lambda \leq D(1 - 1/\Lambda)$ , where  $D$  is the dimension of the physical space. The log-Poisson domain corresponds to  $\Lambda = 1$  and is limited by  $C \leq 2D$ . Ultra-relativistic and degenerate limits correspond to  $C/\Lambda \rightarrow 0$  and to  $C \rightarrow \infty, C/\Lambda$  finite. The limit  $C/\Lambda \rightarrow \infty$  forms the Newtonian group, also acceptable. The mono-fractal case is the  $n \rightarrow \infty$  limit of all preceding cases. Note that, on this graph, the Lorentz group  $\Lambda = 0$  would appear as the upper limit of the ordinate axis.

4.1.4. *Concavity.* — The function  $\delta\zeta(n)$  is concave only for  $n$  belonging to a certain interval. This is the physically admissible range. The point where the second derivative of  $\delta\zeta$  changes sign is singular. In all cases, the first derivative of the intermittency function  $\delta\zeta(n)$  vs.  $n$  is of constant sign:  $\delta\zeta(n)$  is either strictly increasing, or strictly decreasing. On the other hand, the sign of its second derivative depends on the position with respect to the critical scaling exponent  $n_*$ , characterizing the divergency of  $\delta\zeta(n)$ ; results are presented below.

Physically, the graph of  $\delta\zeta(n)$  vs.  $n$  must be concave, as shown in appendix E: condition (E.3) implies that its second derivative is positive everywhere. As is easy to check in all cases, this constrains  $\Gamma$  and  $C_{\pm}$  to be real, and (E.3) implies (16,17). Reciprocally, if (16, 17) are satisfied, the graph of  $\delta\zeta$  is concave in one of the domain delimited by  $n_*$ : there is a physically admissible solution for  $\delta\zeta$ . We may therefore say that the concavity requirement or the positivity of the invariant codimensions are two conditions which are physically equivalent.

The intermittency function diverges at  $n_*$  in all cases; in the case of a single root, the divergence occurs at infinity. We can therefore interpret  $n_*$  as a critical exponent characteristic of the system, which is a physical upper/lower bound of the set of allowed scaling exponents. If we then always constrain the exponents to lay either below or above  $n_*$ , the intermittency function will then be characterized by some well defined concavity properties in the corresponding domain.

4.2. CLASSIFICATION OF THE POSSIBLE STATISTICS. — The shape of the intermittency function determines the scaling exponents of the moments of the fields  $\phi_\ell$ , and hence, its statistics. Indeed, we recall that:

$$\langle \phi_\ell^n \rangle \propto \phi_{\max}^n e^{\delta\zeta(n)}, \tag{29}$$

and:

$$\zeta(n) = n\Delta + \delta\zeta(n). \tag{30}$$

The three possible shapes of  $\delta\zeta$  mentioned in equations (22, 24, 26), together with their special limits, yield six admissible statistics for scale invariant fields (Fig. 2). This classification will be applied e.g. to Burgers equation (Sect. 6) and to turbulence [3].

• General case  $\Lambda > 1$ : if, for given experimental conditions, two types of intermittent structures are observed, then there must be two invariant exponents, both between 0 and the dimension  $D$  (except in the case of negative dimensions, [10]). Experimental measurements thus fix  $C$  and  $\Lambda$  such that  $0 < C \leq (\Lambda - 1)D$ .

If in a given system there exists only one type of intermittent structure, we have one of the two following cases :

• Log-Poisson case  $\Lambda = 1$  : there is only one single invariant exponent  $C_- = C/2 \leq D$ , determined by experimental conditions, with  $C_+$  disappearing at infinity. The codimension function and the scaling exponents are defined for any  $n$ . The variable  $\ln(\phi/\phi_{\max})$  has a characteristic function:

$$\Phi(n) = \exp[\text{cst}(\beta(1)^n - 1)], \quad (31)$$

where the constant may depend on the scale. Here  $\Phi$  generalizes the characteristic function of a Poisson distribution for  $n$  real instead of integer [11]. Note that  $\delta\zeta(1) > 0$  iff  $\beta(1) < 1$ .

• Degenerate case  $C/\Lambda$  finite,  $C \rightarrow \infty$ : there is a double root  $C_{\pm} = \lim_{C \rightarrow \infty} C/\Lambda \leq D$ .

• Newtonian limit  $C/\Lambda \rightarrow \infty$  and  $C \rightarrow \infty$ . It is thus a singular limit of the generic case, as is reflected in the appearance of new symmetries, namely here the full scale invariance as well as the large scale  $\rightarrow$  small scale symmetry  $\ell \rightarrow 1/\ell$ . This is a system with no intermittent structure at all. Alternatively, it can be conceived as a system with infinite codimension of intermittent structures, which is possible only in a  $D \rightarrow \infty$  limit. The intermittency function is linear:  $\delta\zeta(n) = n\delta\zeta(1)$ .

• Ultra-relativistic limit  $C/\Lambda \rightarrow 0$ . The intermittency function  $\delta\zeta$  is zero. This case includes deterministic fields.

• Mono-fractal  $\delta\zeta \equiv C_{\pm}$ : in the limit where one of the invariant codimensions  $C_{\pm}$  is reached for at least one finite  $n$ , the codimension function is constant. This case reproduces the scaling exponents of a mono-fractal process, which only occurs on a fractal of codimension  $C_{\pm}$ ; see [12] for example of such a model in turbulence. Note that it is the asymptotic  $n \rightarrow \infty$  limit of all five preceding cases.

4.3. CASE OF NON-POSITIVE PHYSICAL FIELDS. — When the physical field is not a strictly positive variable, we can separate its negative and positive parts and obtain:

$$\begin{aligned} \langle \phi_{\ell}^n \rangle &= \langle (\phi_{\ell}^{(+)})^n \rangle + (-1)^n \langle (\phi_{\ell}^{(-)})^n \rangle \\ &= A_{+,n} \ell^{n\Delta_+ + \delta\zeta^+(n)} + (-1)^n A_{-,n} \ell^{n\Delta_- + \delta\zeta^-(n)}, \\ \langle |\phi_{\ell}|^n \rangle &= \langle (\phi_{\ell}^{(+)})^n \rangle + \langle (\phi_{\ell}^{(-)})^n \rangle \\ &= A_{+,n} \ell^{n\Delta_+ + \delta\zeta^+(n)} + A_{-,n} \ell^{n\Delta_- + \delta\zeta^-(n)}. \end{aligned} \quad (32)$$

Here, the  $A_{\pm,n}$  are constants which become irrelevant in the small scales limit. Note that equation (32a) is valid only for integer  $n$ , while (32b) also holds for non integer  $n$ . The scaling exponents of  $\phi$  and  $|\phi|$  are then determined by the leading term, i.e. with smallest exponent. Thus the purely academical zero scale limit yields:

$$\begin{aligned} \ell \rightarrow 0 \quad \langle \phi_{\ell}^n \rangle &= \ell^{\zeta(n)}, \\ \langle |\phi_{\ell}|^n \rangle &= \ell^{\xi(n)}, \end{aligned} \quad (33)$$

with

$$\zeta(n) = \xi(n) = \min_n \left( n\Delta_+ + \delta\zeta^+(n), n\Delta_- + \delta\zeta^-(n) \right). \quad (34)$$

In practice, due to destructive interferences, odd exponents of  $\phi^{2j+1}$  are not as well defined as those of  $|\phi|^{2j+1}$ , and experiments may measure a difference between  $\zeta_{2j+1}$  et  $\xi_{2j+1}$  [13].

Table I. — *Differences with Nottale's analogy of speed relativity.*

	<b>Special relativity</b>	<b>Nottale's theory</b>	<b>Present paper</b>
coordinates	time $t$ space $x$	exponent $\ln \phi_\ell$	$T = \ln \ell$ $X = \ln \phi_\ell$
equivalence	inertial frames	resolutions	reference fields
variable	speed $v$	log-scale $\ln(\ell)$	exponent $V = dX/dT$
absolute limits	light speed $-c, +c$	cosmological and Planck lengths	intermittent codimensions $C_\pm$

## 5. Link with Other Existing Theories

5.1. NOTTALE'S THEORY. — Nottale already developed a relativity theory, insisting on the dependence of physics with resolution [4]. He is then led to work with a principle of "scale (resolution) relativity": the field and the exponent plays the role of the spatial and time coordinate; the log-scale  $\ln(\ell)$  is analogous to the velocity  $v$ , and undergoes Lorentz-like transformations. One important result is the existence of an universal limit scale, which he identifies with the limit of our perceptible Universe, namely the Planck length at low scales, or conversely at large scales the length associated to the inverse of the cosmological constant. His theory is "fundamental" in the sense that it applies to the Universe as a whole, and does not take an equation of evolution as a starting point, but the possible fractal structure of the space time.

In contrast, our relativity theory only rigorously applies in the idealistic case where the scale invariance extends from  $\ell = 0$  to  $\ell = \infty$ , in systems described by effective scale covariant equations, such as hydrodynamical turbulence. This example led us to use a different form of relativity, better suited to describe the statistical properties of random processes. Therefore, both theories are physically well distinct, even though both borrowed their formalism to the same source. Here, it is  $T = \ln(\ell)$  which plays the role of a coordinate invariant by translation (Tab. I); the analogous of the referential is labeled by the exponent  $V_{\mathcal{R}} = dX/dT$ . Equation (20) yields as a limit intermittency the codimension  $C_\pm$ .

A major interest of Nottale's theory is the self-consistent inclusion of possible cut-offs in the space scale. This opens the possibility to describe the breaking of the exact scale-invariance in our model using Nottale's theory.

5.2. LINK WITH THE MULTI-FRACTAL THEORY. — We have discussed in Section 4.4.6 the case of a mono-fractal process, characterized by singularities occurring on a set of codimension  $C_\pm$ , leading to scaling exponents  $\zeta(n)$  with a simple affine form. The multi-fractal processes were introduced by Parisi and Frisch [14] as a generalization of the mono-fractal process allowing for the existence of different sets of singularities with different codimensions. The multi-fractal processes are then characterized by a function  $f(h)$ . This function maps real scaling exponents  $h$  to scaling codimensions  $-f \geq 0$  such that for any  $h$ , the measure  $Q(\ell, h)$

of the set of points on which the process has the scaling exponent  $h$  satisfies:

$$Q(\ell, h) \propto \ell^{-f(h)}, \quad \ell \rightarrow 0. \quad (35)$$

A multi-fractal process can therefore be viewed as a superposition of mono-fractal processes with different codimensions. For each mono-fractal with singularity  $h$  on a set of codimension  $C_{\pm} = -f(h)$ , the scaling exponents  $\zeta(n, h)$  have the affine form  $\zeta(n, h) = nh + f(h)$  (see Sect. 4.4.6). The resulting process is then characterized by scaling exponents given by a Legendre formula:

$$\zeta(n) = \min_h (nh - f(h)). \quad (36)$$

In such case, the function  $\zeta(n)$  is not affine anymore, but curved in a way depending on the function  $f(h)$ , which may be obtained experimentally from the measure of  $\zeta(n)$  by an inverse Legendre transform as

$$f(h) = \min_n (nh - \zeta(n)). \quad (37)$$

It has then become customary to call multi-fractal a process in which the function  $\zeta(n)$  is not affine, and to compute the corresponding codimension function  $-f(h)$  by the inverse Legendre transform (37).

It is then interesting to note that all six classes of scale invariant processes are multi-fractal in the sense that the function  $\zeta(n)$  is curved and that it leads to a non trivial function  $f(h)$  by application of the inverse Legendre transform (37). In fact, this connection is deeper. Indeed, one justification of the use of the multi-fractal theory is via a *local* scaling argument: invariance of the random field by local scale dilation  $\mathbf{x} \rightarrow \lambda\mathbf{x}$ ;  $\phi(\mathbf{x}) \rightarrow \lambda^{h(\mathbf{x})}\phi(\mathbf{x})$ , where  $h(\mathbf{x})$  is a local exponent, characterized by the measure (35), or equivalently a probability distribution function. This local scale symmetry can then be seen as invariance by multiplication by an arbitrary random field  $\ell^{h(\mathbf{x})}$ , which is automatically scale invariant, and where all the randomness is in  $h$ . This can be seen as the equivalent of our reference fields  $\mathcal{R}_\ell(\mathbf{x})$ . Our postulate of equivalence then would amounts to *select only the fields  $\ell^{h(\mathbf{x})}$  such that all exponents  $h$  are equivalent*. This additional constraint explains why only a restricted class of multi-fractals is selected.

Note also that all of the six classes are included in the set of infinitely divisible laws. Indeed, the logarithm of a multiplicative scale-invariant process is itself a process with stationary and independent increments. Equivalently, it is described by one of the infinitely divisible laws [15], which includes the normal and Poisson law, and more generally any Levy law.

## 6. Application to Burgers Equation

One of the simplest scale covariant equation is the Burgers equation of hydrodynamical shocks:

$$\partial_t u + u \partial_x u = \nu \partial_{xx} u, \quad (38)$$

where  $t$  is the time,  $u$  the velocity field, and  $\nu$  the viscosity. For smooth initial conditions with a single length scale and a single velocity scale (both taken equal to unity for simplicity), solutions of the Burgers equation display steep variations in the velocity profile, with a width of the order  $O(\nu)$ . In the inviscid limit, these variations steepen into isolated finite discontinuities in the velocity, called shocks [16].

In the inviscid limit, the Burgers equation is covariant under the family of spatial dilations  $\mathcal{S}_h(\lambda)$  with arbitrary similarity exponent  $h$  and scale factor  $\lambda$  [2]:

$$\mathcal{S}_h(\lambda) : \quad x \rightarrow \lambda x, \quad u \rightarrow \lambda^h u, \quad t \rightarrow \lambda^{1-h} t. \quad (39)$$

Statistically stationary regimes of the Burgers equation may be obtained e.g. by adding a random force such that spatial, temporal and amplitude scales are  $O(1)$ . In the “inertial” range of scale  $\nu \ll \ell \ll 1$  they can be studied within the framework of the present theory. For random homogeneous initial conditions with zero mean value, the scaling properties of the solutions can be investigated via the velocity gradients over a distance  $\ell$ :

$$du_\ell(x) = \frac{u(x + \ell) - u(x)}{\ell}. \tag{40}$$

These quantities can be seen as the coarse grained average at scale  $\ell$  of the velocity gradient  $\partial_x u$ .

As discussed in [17], the coexistence of shocks and smooth regions in solutions of the Burgers equation singles out two exponents and two codimensions. For the shocks,  $u(x + \ell) - u(x) \sim O(1)$  is negative, independent of  $\ell$  and occurs at isolated points; this selects  $\Delta_- = -1$  and  $C_- = 1$  for the shocks. In the smooth regions, which occupies most of the space, the velocity is regular, and takes the form of linear ramps going from left to right. This gives positive velocity differences, with a linear scaling i.e.  $u(x + \ell) - u(x) \sim \ell$ , so that  $\Delta_+ = 0$  and  $C_+ = 0$ . The existence of two different codimensions in the problem imposes that the intermittency function in the Burgers model falls in the generic case. Then, taking into account (34), we find the following scaling exponents for the absolute value in the limit  $C_+ \rightarrow 0$ :

$$\begin{aligned} \xi(n) = \zeta_+(n) = 0 & \quad \text{for } 0 < n < 1, \\ \xi(n) = \zeta_-(n) = -n + 1 & \quad \text{for } n > 1. \end{aligned} \tag{41}$$

Coming back to the usual velocity differences  $\delta u_\ell = u(x + \ell) - u(x)$  over a distance  $\ell$ , we obtain a similar scaling law:

$$\langle |\delta u_\ell|^n \rangle \propto \ell^{\xi'(n)}, \tag{42}$$

with

$$\begin{aligned} \xi'(n) = n & \quad \text{for } 0 < n < 1, \\ \xi'(n) = 1 & \quad \text{for } n > 1. \end{aligned} \tag{43}$$

This results coincides with scaling exponents obtained in numerical simulations or theoretical considerations, based e.g. on the multi-fractal theory [17]. In the present context, it is interesting to note that the bifractal shape (43) originates from two effects: the coexistence of negative and positive fluctuations, with different scaling behaviours, and the existence of two invariant codimensions. These two effects are physically well distinct, since only the second one is a generic outcome of our scale invariant theory, i.e. derived from symmetry considerations.

### 7. Conclusion

In this paper, we develop a formalism to determine the possible statistics of random processes in a scale invariant system. Our argumentation is based on the combination of a simple postulate of exponent relativity, based on arbitrary choice of scale coordinates, and some fundamental properties of scale invariant systems: arbitrary choice of the reference scale and homogeneity in log-coordinates. In complete analogy with relativity theory, we may then derive the structure of similarity transformations, linking two systems of scales, provided they involve only regular functions of only one scale at the time. We find that similarity transformations form a one-dimensional group depending on one parameter, the exponent  $V$ .

This group is labeled by two real constants, characteristic of the physical system:  $C$  is a typical exponent of order unity;  $\Lambda$  represents the symmetry-breaking between small and large scales. These constants determine the invariant element of the group and their associated invariant codimension  $C_{\pm}$ . The latter are identified with the codimension of the most intermittent structures in the systems, by analogy with fractal processes.

We are then led to a classification of the possible scale invariant processes, their statistics and their scaling exponents. These processes are characterized by a continuous function  $\delta\zeta(n)$ , called intermittency function, describing the scaling properties of their probability distribution function, and obeying the group composition law. They can be classified into 6 categories, depending on the invariant codimensions  $C_{\pm}$  of the composition law. The generic and degenerate classes are characterized by divergences of moments with order lower or larger than a critical order. The ultra-relativistic and Newtonian classes are their trivial limits. The log-Poisson and mono-fractal classes are regular.

The present theory is then linked to the relativistic mechanics and to Nottale's theory. It also applies to hydrodynamics: first to shocks; then to developed turbulence seen from the point of view of multi-fractal models or, as generalized in [7], using a Lagrangian formalism in the space of scales.

### Acknowledgments

We express our gratitude to A. Pocheau and L. Nottale for discussions around their seminal articles. Since we first submitted this paper, we received a preprint from A. Pocheau in which he explores various consequences of scale gauge invariance [18]. E. Tantart pointed out the axiomatic approach of [8]. We have the pleasure to acknowledge fruitful informal discussions with J. Lajzerowicz, A. Noullez, L. Bréon and S. Graner. This work was supported by a grant from the European Community (ERBCHRXCT920001), and Groupement de Recherche CNRS-IFREMER "Mécanique des Fluides Géophysiques et Astrophysiques".

### Appendix A

#### Postulates

We define a similarity transformation as the transformation which links the coordinates of a given scale invariant homogeneous field with respect to two different reference fields  $\mathcal{R}$  and  $\mathcal{R}'$ . In our log-coordinates, the problem is thus to determine  $(X' - X'_0, T' - T'_0)$  once we know the set of  $(X - X_0, T - T_0)$ .

We explore here the simplest case where all physical variables depend only on one scale. This is an essential point raised by Pocheau [6] who discussed why in turbulent fronts, the correlation length  $\xi$  in the scale space is infinite, so that a velocity at one scale formally depends on the velocities at all other scales. In the case of isotropic turbulence, he suggested that  $\xi$  is zero, and only one scale should be considered. This amounts to writing that the transformation law depends only on one single variable, i.e.  $(X', T')$  is a function of  $(X, T)$  only; this hypothesis is not fundamental and might have to be relaxed for systems other than isotropic turbulence, see the paper [7]. We thus characterize the similarity transformations of intervals by two functions  $F, G$  and *a priori* one [19] transformation parameter  $\alpha$  :

$$\begin{aligned} X' - X'_0 &= F(X - X_0, T - T_0, \alpha) \\ T' - T'_0 &= G(X - X_0, T - T_0, \alpha), \end{aligned} \tag{A.1}$$

where, thanks to Postulate 1, we can impose:

$$F(0, 0, \alpha) = G(0, 0, \alpha) = 0. \tag{A.2}$$

In (A.1), we emphasize the separation into global dilations on one hand, and similarity transformations on the other hand; in these variables they correspond respectively to global and local translations. We now will derive the precise functional form of  $F$  and  $G$  using additional natural assumptions. In particular we assume from now on that  $F$  and  $G$  are both regular and differentiable with respect to their variables.

The next step uses again directly the scale symmetry: *the similarity transformation of an interval  $X_2 - X_1, T_2 - T_1$  depends only on that interval and not of its end points.* This homogeneity holds because scale symmetry amounts to an homogeneity in the log-variables (see Sect. 3). This homogeneity property precludes the existence of any privileged position in the scale space. It can be seen as a generalization, in log-coordinates, of the self-similar property: for a self-similar field  $u_\ell(\mathbf{x})$  the ratio  $u_{\ell_1}(\mathbf{x})/u_{\ell_2}(\mathbf{x})$  depends only on the ratio  $\ell_1/\ell_2$ ; this property is heavily used e.g. in the cascade models of turbulence [12].

Considering an infinitesimal interval  $(dX, dT)$  the most general transformations may be written:

$$\begin{aligned} dX' &= \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial T} dT, \\ dT' &= \frac{\partial G}{\partial X} dX + \frac{\partial G}{\partial T} dT. \end{aligned} \tag{A.3}$$

Homogeneity implies that the coefficients of  $dX$  and  $dT$  must be independent of  $X$  and  $T$ , so that  $F$  and  $G$  are linear functions of  $X$  and  $T$ . Taking into account the condition (A.2), we may then write an homogeneous transformation (A.1) between two frames with the same origin,  $(X'_0, T'_0) = (X_0, T_0)$ :

$$\begin{pmatrix} X' \\ T' \end{pmatrix} = \mathcal{M}(\alpha) \begin{pmatrix} X \\ T \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{XX'} & \mathcal{M}_{TX'} \\ \mathcal{M}_{XT'} & \mathcal{M}_{TT'} \end{pmatrix} \begin{pmatrix} X \\ T \end{pmatrix}, \tag{A.4}$$

where the  $2 \times 2$  matrix  $\mathcal{M}$  only depends on  $\alpha$ . Note that the homogeneity implies the linearity, which does not need to be postulated [4].

Consider now a similarity transformation (A.4) linking the coordinates of a reference field  $\mathcal{R}'$  with respect to another reference field  $\mathcal{R}$  and with respect to itself. In log-coordinates, it amounts to going from  $X = V_{\mathcal{R}'|\mathcal{R}}T$ , into  $X' = 0$  for any  $T'$ . This therefore imposes the shape of the ratio  $\mathcal{M}_{TX'}/\mathcal{M}_{XX'}$  as  $V_{\mathcal{R}'|\mathcal{R}}$ . It seems therefore natural to use  $V$  as the transformation parameter. Changing notation, we now write the general transformation formulae as (B.1).

## Appendix B

### Derivation of Similarity Transformations

We will demonstrate that the only possible expression (A.4) compatible with a semi-group structure is (11), by generalizing Lévy-Leblond [8] and Nottale [4] to the case where no parity restriction is imposed.

From appendix A, we can change notations rewrite (A.4) as:

$$\begin{aligned} X' &= \Gamma(V)[X - VT], \\ T' &= \Gamma(V)[-A(V)X + B(V)T], \end{aligned} \tag{B.1}$$

depending on three yet unknown functions  $\Gamma$ ,  $A$  and  $B$ . The dependence of  $V$  in  $\mathcal{R}'|\mathcal{R}$  is implicit. We apply (B.1) first with a parameter  $V$ , then for  $V'$ , and require that the resulting transformation should be associated to a third parameter, noted  $V'' = V \otimes V'$ ; we thus get the following set of conditions [4]:

$$\begin{aligned} V'' &= \frac{V + B(V)V'}{1 + A(V)V'} \\ A(V'') &= \frac{A(V') + A(V)B(V')}{1 + A(V)V'} \\ B(V'') &= \frac{B(V)B(V') + VA(V')}{1 + A(V)V'} \\ \Gamma(V'') &= \Gamma(V)\Gamma(V')(1 + A(V)V'). \end{aligned} \quad (\text{B.2})$$

Taking  $V = V' = 0$  in (B.2), we find the conditions:

$$\begin{aligned} A(0) &= [1 + B(0)]A(0), \\ B^2(0) &= B(0), \\ \Gamma^2(0) &= \Gamma(0). \end{aligned} \quad (\text{B.3})$$

From (Eqs. (B.3b,c)),  $B(0)$  and  $\Gamma(0)$  are equal to 0 or 1; only  $B(0) = \Gamma(0) = 1$  allows for the existence of nontrivial transformations, including the neutral one (identity). Then, from (Eq. (B.3a)),  $A(0) = 0$  and it is natural to introduce the two new functions  $a(V)$  and  $b(V)$  defined as:

$$A(V) = a(V)V, \quad B(V) = 1 + b(V)V. \quad (\text{B.4})$$

Using these functions, the conditions (B.2) write:

$$\begin{aligned} V'' &= \frac{V + V' + b(V)VV'}{1 + a(V)VV'} \\ a(V'') &= \frac{a(V)V + a(V')V' + a(V)b(V')VV'}{V + V' + a(V)VV'} \\ b(V'') &= \frac{b(V)V + b(V')V' + b(V)b(V')VV'}{V + V' + a(V)VV'} \\ \Gamma(V'') &= \Gamma(V)\Gamma(V')(1 + a(V)VV'). \end{aligned} \quad (\text{B.5})$$

If  $A$  and  $B$ , and thus  $a$  and  $b$ , are continuously derivable functions of  $V$ , the composition law is necessarily commutative, i.e.  $V''(V, V') = V''(V', V)$  [20]. Therefore  $a(V) = a(V')$  and  $b(V) = b(V')$  for all  $V, V'$ . They are constant functions, and we prefer to introduce the two constants  $C, \Lambda$  that we use in the text:

$$\begin{aligned} \frac{1}{C^2} &= a + \frac{b^2}{4}; \\ \frac{1}{\Lambda^2} &= \frac{4a}{b^2} + 1. \end{aligned} \quad (\text{B.6})$$

This proves (11). Since  $a$  and  $b$  are *a priori* arbitrary real numbers, we note that:

- $C$  is never zero; while  $\Lambda = 0$  if  $b = 0$ ;
- if  $4a + b^2 < 0$ , then both  $C$  and  $\Lambda$  are imaginary numbers;
- if  $4a + b^2 > 0$ , then both  $C$  and  $\Lambda$  are real; since (11) is invariant under the choice  $C \rightarrow -C, \Lambda \rightarrow -\Lambda$ , we can arbitrarily decide to choose  $C > 0$ .



Here  $\Lambda$  and  $C$  are yet undetermined; either they are both real numbers, or they are both purely imaginary;  $C$  is never zero (see appendix B). Because of the symmetry:

$$\Lambda \rightarrow -\Lambda, \quad C \rightarrow -C, \tag{B.7}$$

we may restrict ourselves to cases where  $C$  has a positive real part:

$$\Re(C) \geq 0. \tag{B.8}$$

**Appendix C**

**Derivation of  $\Gamma$**

To find  $\Gamma$  we now note that for all  $V, V'$  and  $V''$  given by (B.5a), we have:

$$(1 + aVV')^2(1 + bV'' - aV''^2) = (1 + bV - aV^2)(1 + bV' - aV'^2). \tag{C.1}$$

We then define  $g$  (possibly complex) as:

$$\Gamma(V) = g(V)(1 + bV - aV^2)^{-1/2} = g(V) \left[ 1 - 2\Lambda \frac{V}{C} + (\Lambda^2 - 1) \frac{V^2}{C^2} \right]^{-1/2}. \tag{C.2}$$

Using (B.5d) and recalling that  $\Gamma(0) = 1$  we obtain:

$$\begin{aligned} [g(V'')]^2 &= [g(V)g(V')]^2, \\ g(0) &= 1, \end{aligned} \tag{C.3}$$

for all  $V'', V'$  and  $V$  satisfying (B.5a). There is an obvious solution  $g(V) = 1$ . When is it the only one ?

It can be shown that it is the only solution when  $\Lambda$  is real and different from 1. Then the function  $k(V)$ , given by:

$$k(V) = V^{[2]} = V \otimes V = \frac{2V(1 - \Lambda V/C)}{1 - (\Lambda^2 - 1)V^2/C^2}, \tag{C.4}$$

has two (possibly equal) stable fixed points  $C_{\pm}$  solutions of (Eq. (D.1)):  $C_{\pm} = k(C_{\pm})$ . They satisfy  $[g(C_{\pm})]^2 = 1$ , while the neutral  $V = 0$  is an unstable fixed point of  $k$ . In that case, by considering the serie  $V_n = k(V_{n-1})$ , converging towards a fixed point for any  $V_0$  in the neighborhood of 0, we find that  $g(V_n) = [g(V_0)]^{2^n}$ , and so  $g(V_0) = 1$ . Continuity of  $g$  is then sufficient to ensure that  $g(C_{\pm}) = 1$ , then that  $g$  is equal to 1 everywhere. To summarize, the group structure ensures that (11) has a determinant 1 so that after repeated similarity transformations  $X$  and  $T$  remain finite. This achieves to establish (11).

On the opposite, when  $\Lambda$  is imaginary (see Sect. 4.1 for discussion) or  $\Lambda = 1$  there are other acceptable solutions, beside  $g = 1$ . For instance, when  $\Lambda = 1$ , the composition law for  $\Gamma$  reduces to a simple multiplication:

$$\Gamma(V \otimes V') = \Gamma(V)\Gamma(V'). \tag{C.5}$$

For any real number  $n$ , there is an acceptable group:

$$\begin{aligned} g(V) &= (1 - 2V/C)^n, \\ X' &= (1 - 2V/C)^{n-1/2}[X - VT], \\ T' &= (1 - 2V/C)^{n+1/2}T, \\ V \otimes V' &= V + V' - 2VV'/C, \end{aligned} \tag{C.6}$$

which has an inverse:

$$\begin{aligned} V^{[-1]} &= -V/(1 - 2V/C), \\ X &= (1 - 2V/C)^{-n-1/2}[(1 - 2V/C)X - VT], \\ T &= (1 - 2V/C)^{-n+1/2}T. \end{aligned} \quad (\text{C.7})$$

In fact, they all amount to the group structure of (11), which is the simplest one, with  $n = 0$  and a determinant 1.

## Appendix D

### Invariant Codimensions

The group structure is entirely determined by the values of  $V$  such as  $\Gamma^{-2}(V) = 0$ . They are the absorbent elements of the composition law  $\otimes$  and are the solutions of the equation:

$$(\Lambda^2 - 1)\frac{V^2}{C^2} - \Lambda\frac{V}{C} + 1 = 0, \quad (\text{D.1})$$

For reasons which appear in Section 3.3, we call them the “invariant codimensions”:

$$\begin{aligned} C_- &= \frac{C}{\Lambda + 1} & ; & & C_+ &= \frac{C}{\Lambda - 1}, \\ \Lambda &= \frac{C_+ + C_-}{C_+ - C_-} & ; & & C &= \frac{2C_+ C_-}{C_+ - C_-}. \end{aligned} \quad (\text{D.2})$$

They have the following properties:

$$\begin{aligned} C_{\pm} \otimes V &= C_{\pm}, \quad \forall V \neq C_{\mp}, \\ C_{\pm}^{[-1]} &= C_{\mp}. \end{aligned} \quad (\text{D.3})$$

For  $C^2$  negative, i.e.  $C$  and  $\Lambda$  imaginary numbers,  $C_-$  and  $C_+$  are complex. When  $C$  is real, three cases appear:

- For  $\Lambda^2 - 1 > 0$ ,  $C_-$  and  $C_+$  have the sign of  $\Lambda$ . If  $\Lambda/C > 0$ , the composition law  $\otimes$  conserves the interval  $I = [0, C_-]$ ; similarly, if  $\Lambda/C < 0$ , the interval  $I = [C_-, 0]$  is conserved. Note that in the limit  $C \rightarrow \infty$ ,  $\Lambda \rightarrow \infty$ ,  $C/\Lambda \rightarrow C_0$ , the invariant codimensions become equal to the double root  $C_+ = C_- = C_0$ .
- For  $\Lambda = 1$ , equation (D.1) becomes a first degree equation: its only solution is  $C_- = C/2$ , while  $C_+$  diverges; the conserved interval is  $I = [0, C_-]$ . The case  $\Lambda = -1$  is similar.
- For  $\Lambda^2 - 1 < 0$ ,  $C_-$  and  $C_+$  have an opposite sign. The law  $\otimes$  conserves the intervals  $[C_-, 0]$ ,  $[0, C_+]$  as well as their union. The group structure is then analogous to the Lorentz group, which we obtain exactly when  $\Lambda = 0$ .

## Appendix E

### Concavity of the Intermittency Function

The scaling exponents and the intermittency function can be proved to be necessarily concave thanks to the Hölder inequality for moments of random variables. This inequality states that for any random variable  $\Phi$  and  $\Phi'$ , and for any integer  $p, q$  such that  $p^{-1} + q^{-1} = 1$ , we have [21]:

$$\langle \Phi \Phi' \rangle \leq \langle \Phi^p \rangle^{1/p} \times \langle \Phi'^q \rangle^{1/q}. \quad (\text{E.1})$$

Taking  $\Phi = \phi^{n/p}$  and  $\Phi' = \phi^{n'/q}$ , where  $\phi$  is a scale invariant field following:  $\langle \phi_\ell(x)^n \rangle = A_n \ell^{\zeta(n)}$ ; this inequality writes, using the constants  $A$  defined as in equation (32):

$$A_{n/p+n'/q} \ell^{\zeta(n/p+n'/q)} < A_n^{1/p} \ell^{\zeta(n)/p} A_{n'}^{1/q} \ell^{\zeta(n')/q}. \quad (\text{E.2})$$

Taking the limit  $\ell \rightarrow 0$ , we then find:

$$\zeta\left(\frac{n}{p} + \frac{n'}{q}\right) \geq \frac{1}{p}\zeta(n) + \frac{1}{q}\zeta(n'), \quad (\text{E.3})$$

which expresses that the graph of  $\zeta$  vs.  $n$  is concave. A similar proof can also be given for  $\delta\zeta(n)$ .

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- [19] Proof: if there were two parameters  $\alpha, \beta$  (or more), for any arbitrary log-coordinates  $(X, T)$  and  $(X', T')$  it would be generically possible to find a solution  $\alpha, \beta$  of equation (A.1), ie an interval could have arbitrary coordinates; on the other hand, without any parameter there would also be no interesting physics. More generally, in a  $p$ -dimensional

space, there can be  $p - 1$  free parameters coupling the  $p$  independent basis coordinates. For less than  $p - 1$ , the independence of the coordinates is not fully exploited, and if there were  $p$  free parameters or more it would mean that the coordinates do not form a complete basis. For a more detailed discussion see [8].

- [20] Proof: the group structure is then associated to the only one-dimensional Lie group, which is the same as for the set  $\mathbf{R}$  of real numbers; see e.g. Chevalley C., "Theory of Lie Groups" (Princeton University Press, 1946). This is also easy to check in a pedestrian way for any  $V$ , starting from  $V' = 0$  by successive infinitesimal increments.
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