LETTERS AND COMMENTS

Comment on 'About the magnetic field of a finite wire'*

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Abstract
A flaw is pointed out in the justification given by Charitat and Graner (2003 *Eur. J. Phys.* **24** 267) for the use of the Biot–Savart law in the calculation of the magnetic field due to a straight current-carrying wire of finite length.

Charitat and Graner (CG) [1] apply the Ampère and Biot–Savart laws to the problem of the magnetic field due to a straight current-carrying wire of finite length, and note that these laws lead to different results. According to the Ampère law, the magnitude B of the magnetic field at a perpendicular distance r from the midpoint along the length $2l$ of a wire carrying a current I is $B = \mu_0 I/(2\pi r)$, while the Biot–Savart law gives this quantity as $B = \mu_0 I l/(2\pi r \sqrt{r^2 + l^2})$ (the right-hand side of equation (3) of [1] for B cannot be correct as $\sin\alpha$ on the left-hand side there must equal $l/\sqrt{r^2 + l^2}$). To explain the fact that the Ampère and Biot–Savart laws lead here to different results, CG say that the former is applicable only in a time-independent magnetostatic situation, whereas the latter is 'a general solution of Maxwell–Ampère equations'. A straight wire of finite length can carry a current only when there is a current source at one end of the wire and a current sink at the other end—and this is possible only when there are time-dependent net charges at both ends of the wire. These charges create a time-dependent electric field and thus the problem is outside the domain of magnetostatics (we note that the time-dependent flux of this electric field is used in equation (7) of [1] with the incorrect sign; the desired result for the magnetic field can be obtained there simply by using the fact that $\partial q(t)/\partial t = -I$).

We would like to point out that both the Coulomb and Biot–Savart laws have to be generalized to give correctly the electric and magnetic fields due to general time-dependent charge and current sources $\rho(r, t)$ and $J(r, t)$. These generalizations are given by Jefimenko's equations [2–4]:

$$E(r, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \left[\frac{\rho(r', t')}{R^2}\hat{R} + \frac{1}{cR}\frac{\partial \rho(r', t')}{\partial t'}\hat{R} - \frac{1}{c^2 R}\frac{\partial J(r', t')}{\partial t'} \right]_{t'=t-R/c} \quad (1)$$

$$B(r, t) = \frac{\mu_0}{4\pi} \int d^3r' \left[\frac{J(r', t')}{R^2} + \frac{1}{cR}\frac{\partial J(r', t')}{\partial t'} \right]_{t'=t-R/c} \times \hat{R} \quad (2)$$

where $R = |r - r'|$, $\hat{R} = (r - r')/R$ and $t' = t - R/c$ is the retarded time. Equations (1) and (2) are the exact causal (retarded) solutions to Maxwell's equations for given time-dependent charge and current densities $\rho(r, t)$ and $J(r, t)$. They can be obtained from the retarded scalar and vector potentials

$$V(r, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(r', t - R/c)}{R} \qquad A(r, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{J(r', t - R/c)}{R} \qquad (3)$$

of the well-known Lorentz-gauge solution to Maxwell's equations by performing the spatial differentiations in

$$E(r, t) = -\nabla V(r, t) - \partial A(r, t)/\partial t \qquad B(r, t) = \nabla \times A(r, t) \qquad (4)$$

with due regard for the implicit dependence of the quantities $\rho(r', t - R/c)$ and $J(r', t - R/c)$ on the coordinate r through the retarded time $t - |r - r'|/c$ [4]. A comparison of equation (3) with equations (1) and (2) shows that, while the Lorentz-gauge potentials can be obtained by a straightforward 'retarding' of the electrostatic scalar and magnetostatic vector potentials, the general expressions for the fields directly in terms of the sources contain terms that are additional to those of the simply 'retarded' Coulomb and Biot–Savart laws.

Jefimenko's equation (2) for the magnetic field reduces to the standard Biot–Savart law only when one can replace the retarded time t' in the integrand with the present time t and drop the term with the time derivative of the current density. Obviously, this can be done when the current density is constant in time but, nontrivially, also when the current density at the retarded time t' is given sufficiently accurately by the first-order Taylor expansion about the present time t (see [4], problem 10.12):

$$J(r', t') \approx J(r', t) + (t' - t)\frac{\partial J(r', t)}{\partial t} = J(r', t) - \frac{R}{c}\frac{\partial J(r', t)}{\partial t}. \qquad (5)$$

The Biot–Savart solution to the problem under discussion is thus correct when the condition (5) is satisfied, as, for example, when the current I is constant in time or varies with time only linearly. However, CG's contention that the standard Biot–Savart law is a general solution of the Maxwell–Ampère equation $\nabla \times B = \mu_0 J + c^{-2} \partial E/\partial t$ is incorrect. Contrary to an assertion of CG, the curl of a non-retarded vector potential (the r^2 in equation (8) of [1] for this quantity is obviously a mistake) does not result in a magnetic field that satisfies that equation in general. Only the curl of the retarded vector potential, which yields Jefimenko's generalization (2) of the Biot–Savart law:

$$\frac{\mu_0}{4\pi}\nabla \times \int d^3r' \frac{J(r', t - R/c)}{R} = \frac{\mu_0}{4\pi} \int d^3r' \left[\frac{J(r', t')}{R^2} + \frac{1}{cR}\frac{\partial J(r', t')}{\partial t'} \right]_{t' = t - R/c} \times \hat{R} \qquad (6)$$

is, together with Jefimenko's generalization (1) of the Coulomb law, the solution of Maxwell's equations with general time-dependent sources.

CG have expressed an agreement with the substance of this comment [5].

References

[1] Charitat T and Graner F 2003 About the magnetic field of a finite wire *Eur. J. Phys.* **24** 267–70
[2] Jefimenko O D 1966 *Electricity and Magnetism* (New York: Appleton-Century-Crofts)
    Jefimenko O D 1989 *Electricity and Magnetism* 2nd edn (Star City, WV: Electret Scientific)
[3] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
[4] Griffiths D J 1999 *Introduction to Electrodynamics* 3rd edn (Englewood Cliffs, NJ: Prentice-Hall)
[5] Charitat T and Graner F 2003 personal communication